PMM U.S.S.R.,Vol.47,No.1,pp.126-129,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 539.37

INVERSE PROBLEM OF PLANE PLASTICITY THEORY*

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A method is proposed to determine the plastic deformation according to a law, given beyond the elastic limit, for the variation in the state of stress. On the basis of the equation obtained, inelastic deformations are calculated for the problems of L.A. Galin /1/ and G.P. Cherepanov /2/ on the extension of an infinite plane with a circular hole.

1. Initial representations. Elastic deformation is generated because of a change in the interatomic spacing, while inelastic deformation is the result of a change in the order of atoms arrangement in the body by slipping over the interatomic planes. In pure form inelastic deformation (without elastic) cannot occur in a real solid. Quantitatively, the plastic deformation tensor Γ_{jk} $(j,\ k = x,\ y,\ z)$ is defined as the difference between the total deformation tensor and its elastic portion. The elastic deformation is here related to Hooke's law with stress tensor components.

It has been shown /3/ that the state of stress due to a given plastic deformation in the plane case (on the boundary of the plasticity domain the inelastic deformation is zero) can be represented as the stress components due to wedgelike dislocations distributed over the plasticity domain with density p, and over the boundary L of the plasticity domain with density p_L

$$p = \frac{1}{2}\frac{\partial^2\Gamma_{xx}}{\partial y^2} + \frac{1}{2}\frac{\partial^2\Gamma_{yy}}{\partial x^2} - \frac{\partial^2\Gamma_{xy}}{\partial x\,\partial y} + \chi \tag{1.1}$$

$$p_L = \frac{1}{2}\left[\left(\frac{\partial\Gamma_{xy}}{\partial y} - \frac{\partial\Gamma_{yy}}{\partial x}\right)\cos(nx) + \left(\frac{\partial\Gamma_{xy}}{\partial x} - \frac{\partial\Gamma_{xx}}{\partial y}\right)\cos(ny)\right] + \chi_L \tag{1.2}$$

where for plane deformation

$$\chi = \frac{\nu}{2}\Delta\Gamma_{zz},\quad \chi_L = -\frac{\nu}{2}\frac{\partial\Gamma_{zz}}{\partial n}.$$

and for the plane state of stress

$$\chi = 0,\ \chi_L = 0$$

Here G is the shear modulus, ν is the Poisson's ratio, n is the external normal to the line L, and Δ is the two-dimensional Laplace operator.

The following results from the above. To find the state of stress originating from the plastic deformations given in an arbitrary body, it is necessary to determine the stress due to one wedgelike dislocation and to sum them. In the case of a multiconnected body, the state of stress due to wedgelike dislocations arranged on each contour that increases its connectedness, is independent of their distribution law but is determined by their total intensity. This results in the concept of the Volterra dislocation.

Therefore, the state of stress can be found that originates in a body for a given plastic deformation. The inverse problem, of determining the plastic deformation by a law of the variation in the state of stress given beyond the elastic limit, is considered below. The plastic deformation in the body should here be such that the state of stress (evaluated by the method elucidated above) due to it and to external forces should agree at each instant with given stress components.

2. Auxiliary problem. Let it be required to find structural imperfections (the density of wedgelike dislocations) over a known field of stress. To solve this problem, we cut a closed rectangular contour in the shape of a frame out of an unloaded body. Then we cut this frame along some section. If there had been defects within the contour then the section

*Prikl.Matem.Mekhan.,Vol.47,No.1,pp.158-162,1983 of the frame would have been separated by an angle which would equal the total power of the wedgelike dislocations included in the contour under consideration. On the other hand, this angle can be found by the elastic deformations of the frame by assuming that they are determined because of an ideal experiment for cutting a body into sufficiently fine parts. Then the density of the wedgelike dislocations, expressed in terms of the stress components, will have the following form in the plasticity domain (for the plane state of stress $\varkappa = (3-\nu)/(1+\nu)$, and for plane deformation $\varkappa = 3-4\nu$)

$$p = -\frac{1+\varkappa}{8G}\Delta(\sigma_x + \sigma_y) \tag{2.1}$$

The density of the wedgelike dislocations on the domain boundary equals /3/

$$p_L = -\frac{1+\varkappa}{8G}\left[\frac{\partial(\sigma_x+\sigma_y)}{\partial n}\right] \tag{2.2}$$

The square brackets denote discontinuity of the enclosed quantity on the boundary of the slip domain.

3. Fundamental problem. It is to determine the plastic deformations in structural imperfections for a given law of stress component variation.

Giving the history of the origination of the state of stress permits determination of the slip line at each instant. Using one of the plasticity theories here, the direction of maximum inelastic shear Γ_m can be found. Then, for a known density of the distributed wedgelike dislocations, the expressions (1.1) and (1.2) can be considered as differential equations relatively to the quantity Γ_m. Their solution is shown in examples (Sects.4 and 5).

The question of the maximal plastic shear does not occur in ideal plasticity problems. In particular, under the Tresca plasticity condition the shear Γ_m agrees with the line of maximal tangential stress action. For such a plasticity condition Galin /1/ for plane deformation and Cherepanov /2/ for the plane state of stress found a hole of radius R under the action of forces q_1, q_2 applied at infinity. For these problems approximate values of the displacement components have been obtained /4/ by the small parameter method.

4. Galin's problem. An ellipse whose exterior is mapped conformally on the exterior of a unit circle by the function (z and ζ are complex variables, and τ_T is the yield point)

$$z = b\left(\zeta + \frac{\lambda}{\zeta}\right), \quad b = R\exp\left(\frac{q_1+q_2}{4\tau_T} - \frac{1}{2}\right), \quad \lambda = \frac{q_2-q_1}{2\tau_T} \tag{4.1}$$

is the boundary between the elastic and plastic domains.

We find the density of the wedgelike dislocations which, together with the external load, cause the state of stress of this elastic-plastic problem in an elastic body, using formulas (2.1) and (2.2).

It can be confirmed that there are no structural imperfections distributed over the plasticity domain, i.e.,

$$p = 0 \tag{4.2}$$

The density of the wedgelike dislocations inserted on the boundary of the plasticity domain equals (the derivative with respect to the normal is evaluated on the boundary of the slip domain)

$$p_L = \frac{2(1-\nu)\tau_T}{G}\frac{\partial|\zeta|}{\partial n} \tag{4.3}$$

Besides these imperfections, wedgelike dislocations are formed on the outline of the circular hole, and are reduced to rings with the angular divergence

$$\alpha = -4\pi\tau_T(1-\nu)/G \tag{4.4}$$

Taking into account that the direction of the maximum plastic shear makes the angle $\pi/4$ with the polar radius, we obtain on the basis of the relationships (1.1) and (4.2) (r, θ are polar coordinates)

$$\frac{\partial^2\Gamma_m}{\partial r^2} + 3\frac{\partial\Gamma_m}{r\,\partial r} - \frac{\partial^2\Gamma_m}{r^2\,\partial\theta^2} = 0 \tag{4.5}$$

The equations of the slip lines ξ, η have the form

$$\xi = \ln r + \theta, \quad \eta = \ln r - \theta \tag{4.6}$$

We write the relationship (4.5) in the ξ, η coordinate system

$$2\frac{\partial^2\Gamma_m}{\partial\xi\,\partial\eta}+\frac{\partial\Gamma_m}{\partial\eta}+\frac{\partial\Gamma_m}{\partial\xi}=0 \tag{4.7}$$

The equation obtained is of hyperbolic type. Its solution can be represented by the Riemann /5/ formula as follows (v is the Riemann function, and L_{12} is an arc of the slip domain boundary between two slip lines passing through the point ξ, η):

$$\Gamma_m(\xi,\eta)=\int_{L_{12}} v(\xi,\eta,\xi_0,\eta_0)\,p_L(\xi_0,\eta_0)\,dL \tag{4.8}$$

Here the expression (1.2) has been used.

For equation (4.7) the Riemann function has the form (I_0 is the zero under Bessel function of imaginary argument)

$$v(\xi,\eta,\xi_0,\eta_0)=\exp\left[-\frac{1}{2}(\eta-\eta_0+\xi-\xi_0)\right]I_0(y)$$
$$y=[(\xi-\xi_0)(\eta-\eta_0)]^{1/2}$$

On the basis of the formula for the density of wedgelike dislocations (4.3), we represent the integral (4.8) in the form

$$\Gamma_m(r,\theta)=-\frac{2\tau_T(1-\nu)}{Gb^2(1-\lambda^2)r}\int_{\theta_1}^{\theta_2} r_0^3(\theta_0)I_0\left(\left[\ln^2\frac{r_0(\theta_0)}{r}-(\theta-\theta_0)^2\right]^{1/2}\right)d\theta_0 \tag{4.9}$$
$$r_0(\theta)=b(1-\lambda^2)[1-2\lambda\cos 2\theta+\lambda^2]^{-1/2}$$
$$\theta_{1,2}=\theta-\ln\frac{r_0(\theta_{1,2})}{r}$$

The solution found will be valid if the plastic deformation grows monotonically at each point. Such a process is possible if the slip line has one point of intersection with the ellipse (4.1). The constraint on the load relationship

$$\frac{q_2-q_1}{2\tau_T}\leqslant\sqrt{2}-1$$

follows from this condition.

Evaluating the integral (4.9) for the axisymmetric problem ($q_2=q_1$) we obtain

$$\Gamma_m(r)=\frac{2\tau_T(1-\nu)}{G}\left(1-\frac{b^2}{r^2}\right)$$

The displacement components are easily evaluated if the total deformations are known which equal the sum of the plastic and elastic deformations, i.e., if plastic deformations have been found, then determination of the shifts reduces just to the evaluation of quadratures. The displacement components obtained in this manner for the deformations (4.9) agree with the displacements determined in /6/.

5. Cherepanov's problem. In constrast to plane deformation, in this case the density of the wedgelike dislocations distributed over the plasticity domain is not zero and is determined by formula (2.1)

$$p=\frac{\tau_T R}{(1+\nu)Gr^3} \tag{5.1}$$

The density of the structural imperfections distributed on the boundary of the slip domain, and the angle of divergence of the ring dislocation

$$p_L=-\frac{c\tau_T(4+3a^2)}{(1-\nu)G[4+a^2+2a(\bar\zeta+\zeta)]}\frac{\partial|\zeta|}{\partial n};\quad \alpha=-\frac{2\pi\tau_T}{(1+\nu)G} \tag{5.2}$$
$$c=\frac{q_1+q_2-4\tau_T}{2\tau_T}\qquad\left(a^3+4a+\frac{8(q_2-q_1)}{q_1+q_2-4\tau_T}=0\right)$$

are found analogously to the preceding problem.

The parameter a is the real root of the cubic equation displayed in the parentheses.

On the basis of relationships (1.1) and (5.1), we obtain a differential equation relatively to the maximal inelastic shear

$$\frac{2}{r}\frac{\partial\Gamma_m}{\partial r}+\frac{\partial^2\Gamma_m}{\partial r^2}=\frac{2\tau_T R}{G(1+\nu)r^3}$$

Taking into account that $\Gamma_m=0$ on the boundary of the plasticity domain, we write the solution of the last equation in the form

$$\Gamma_m(r,\theta)=f(\theta)\left(\frac{1}{r}-\frac{1}{r_0}\right)-\frac{2\tau_T R}{G(1+\nu)}\left(\frac{1}{r}\ln\frac{r}{R}-\frac{1}{r_0}\ln\frac{r_0}{R}\right) \tag{5.3}$$

$$r_0=R\frac{4+a^2+4a\cos 2t}{c(a^2-4)}\qquad\left(\operatorname{tg}\theta=\frac{4(1-a)\sin t-a^2\sin 3t}{4(1+a)\cos t+a^2\cos 3t}\right)$$

Here r_0 is the parametric equation of the slip domain boundary, where the parameter t is related to the polar angle θ by the relationship displayed in parentheses.

We determine the arbitrary function $f(\theta)$ from the condition that the density of the wedge-like dislocations on the boundary of the plasticity domain, which is expressed in terms of the deformation (5.3) by means of formula (1.2), should equal the density (5.2) found. We then finally obtain

$$\Gamma_m(r,\theta)=\frac{2\tau_T}{G(1+\nu)}\left\{\left(1-\frac{r_0}{r}\right)\left[\frac{c(4+3a^2)}{4-3a^2-4a\cos 2t}+\frac{R}{r_0}\right]-\frac{R}{r}\ln\frac{r}{r_0}\right\}$$

The solution considered will be valid while the plasticity domain encloses a circular hole and is here broadened monotonically; moreover, there should be $0\leqslant a\leqslant {}^2/_3$.

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Tranlated by M.D.F.